ON THE STABILITY OF MOTION OF A GYROSCOPE ON GIMBALS IN A CONSERVATIVE FORCE FIELD

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The problem of motion and of stability of a heavy symmetric gyroscope on gimbals has been investigated in [1-4], where the necessary and sufficient conditions of stability of the stationary motion were obtained by the second method of Liapunov. In this paper the author presents similar results for a gyroscope on gimbals in a force field determined by a force function $V(\theta)$, where θ is the rotation angle of the inner ring (casing). This problem has been solved in the first approximation for a gyroscope not on gimbals and reported in [5].

Considering a symmetric gyroscope on gimbals (see figure), we shall introduce the following symbolism: x_1, y_2, z_3 is the fixed coordinate system, x, y, z is the moving coordinate system rigidly connected with the casing (the x-axis along the axis of the casing, the z-axis along the rotor's axis); ψ is the rotation angle of the outer ring: ϕ is the rotation angle of the gyroscope; A, A, C are the moments of inertia of the gyroscope about x-, y-, z-axes respectively; A_1, B_1, C_1 are the moments of inertia of the casing about the same axes; J is the moment of inertia of the outer ring about the vertical axis z_1.

The equations of motion of our system in the Lagrange form

$$(A+A_1)\ddot{\theta}-(A+B_1-C_1)\dot{\psi}^2\sin\theta\cos\theta+C(\dot{\varphi}+\dot{\psi}\cos\theta)\dot{\psi}\sin\theta=-V'(\theta)$$

$$\frac{d}{dt}\left\{[(A+B_1)\sin^2\theta+C_1\cos^2\theta+J]\dot{\psi}+C(\dot{\varphi}+\dot{\psi}\cos\theta)\cos\theta\right\}=0 \qquad (1)$$

$$C\frac{d}{dt}(\dot{\varphi}+\dot{\psi}\cos\theta)=0$$

admit the three first integrals

$$[(A+B_1)\sin^2\theta + C_1\cos^2\theta + J]\,\dot\psi^2 + (A+A_1)\,\dot\theta^2 + C\,(\dot\varphi + \dot\psi\cos\theta)^2 + 2V(\theta) = h \qquad (2)$$

$$[(A+B_1)\sin^2\theta + C_1\cos^2\theta + J]\,\dot\psi + C\,(\dot\varphi + \dot\psi\cos\theta)\cos\theta = k \qquad (3)$$

$$r = \dot\varphi + \dot\psi\cos\theta = r_0 \qquad (4)$$

The dot indicates the time derivative, prime denotes a derivative with respect to θ. By separating the variables the solution of the problem can be reduced to quadratures

$$\dot\theta^2 = \frac{[\alpha - mV(\theta)](\varepsilon - e\cos^2\theta) - (\beta - br_0\cos\theta)^2}{\varepsilon - e\cos^2\theta} = \frac{f(\theta)}{\varepsilon - e\cos^2\theta},$$

$$\dot\psi = \frac{\beta - br_0\cos\theta}{\varepsilon - e\cos^2\theta}, \qquad \dot\varphi = r_0 - \dot\psi\cos\theta$$

Here

$$\alpha = \frac{h - Cr_0^2}{A+A_1}, \quad \varepsilon = \frac{A+B_1+J}{A+A_1}, \quad e = \frac{A+B_1-C_1}{A+A_1}, \quad \beta = \frac{k}{A+A_1}$$

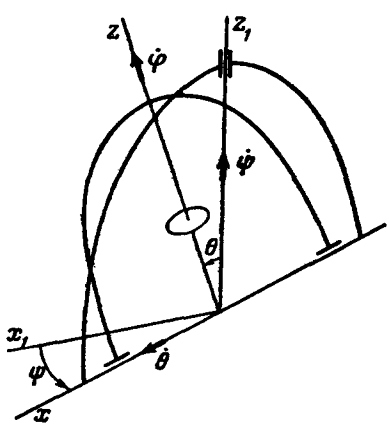
$$b = \frac{C}{A+A_1}, \quad m = \frac{2}{A+A_1}$$

For a heavy gyroscope on gimbals with a vertical axis of the outer ring ($V = a\cos\theta$) solutions lead to hyperbolic integrals which were obtained in [1].

The following special case is of interest:

$$V(\theta) = n/(\epsilon - e\cos^2\theta); \quad (n \text{ is a constant})$$

In this case the function $f(\theta)$ reduces to a second degree polynomial in $u = \cos\theta$. This polynomial differs only by the free term from a corresponding polynomial arising in the case of an equilibrated gyroscope ($V(\theta) \equiv 0$). Investigations of the stability of stationary motions (vertical rotation and regular precession) of a gyroscope can be carried out [1] like for an equilibrated gyroscope, by considering the roots of the polynomial $f(u)$. We shall investigate the stability of the stationary motion of a gyroscope with an arbitrary analytic function $V(\theta)$.



The regular precession of the gyroscope

$$\theta = \theta_0, \qquad \dot\theta = 0, \qquad \dot\psi = \Omega, \qquad r = \omega \qquad (5)$$

occurs, according to (1), when the constants θ_0, Ω, ω satisfy the relation

$$(A+B_1-C_1)\,\Omega^2\sin\theta_0\cos\theta_0-C\omega\Omega\sin\theta_0-V'(\theta_0)=0 \tag{6}$$

In order to find the stability of the solutions (5) with respect to θ, $\dot{\theta}$, $\dot{\psi}$, r, we shall write down the equations of the perturbed motion by substituting

$$\theta=\theta_0+\eta,\qquad \dot{\theta}=\xi_1,\qquad \dot{\psi}=\Omega+\xi_2,\qquad r=\omega+\xi_3$$

The equations of the perturbed motion permit three integrals which correspond to the integrals (2), (3), (4). These integrals can be combined to form a function similar to the one shown in [2]. Conditions for sign-definiteness of this function with respect to all the variables η, ξ_1, ξ_2, ξ_3 reduce to one inequality

$$(A+B_1-C_1)\,\Omega^2\cos 2\theta_0-C\omega\Omega\cos\theta_0-V''(\theta_0)<0 \tag{7}$$

which is the sufficient condition of stability of the motion (5) with respect to θ, $\dot{\theta}$, $\dot{\psi}$, ψ, r. In the case when $\theta_0\neq 0$, this condition could be expressed in the form

$$(A+B_1-C_1)\,\Omega^2\sin^2\theta_0-V'(\theta_0)\cot\theta_0+V''(\theta_0)>0 \tag{8}$$

by using Equation (6).

For the vertical rotation, $\theta_0=0$, the conditions (6), (7), take the form

$$V'(0)=0,\qquad (A+B_1-C_1)\,\Omega^2-C\omega\Omega-V''(0)<0 \tag{9}$$

The inequality (9) will be satisfied when the following inequalities are satisfied

$$C^2\omega^2+4\,(A+B_1-C_1)\,V''(0)>0,\qquad \Omega_1<\Omega<\Omega_2 \tag{10}$$

where Ω_1, Ω_2, are roots of the quadratic equation

$$(A+B_1-C_1)\,\Omega^2-C\omega\Omega-V''(0)=0$$

It can be easily shown that the condition (9) is also the necessary condition of stability of vertical rotations of a gyroscope. On the strength of the perturbed equations

$$W=(A+A_1)\,\eta\xi_1$$

the time derivative of the function

$$\frac{dW}{dt} = (A + A_1)\xi_1^2 + [(A + B_1 - C_1)\Omega^2 - C\omega\Omega - V''(0)]\eta^2 + \ldots$$

is positive-definite when the following inequality is satisfied:

$$(A + B_1 - C_1)\Omega^2 - C\omega\Omega - V''(0) > 0$$

The motion under consideration is, by Liapunov's theorem, unstable.

This means that the condition (9) is the necessary and sufficient condition (excluding the boundary) of stability of the vertical rotations of a gyroscope on gimbals.

For the regular precessions, $\theta_0 \neq 0$, we can also obtain the necessary and sufficient conditions of stability by using the theorem of Routh. The changed potential energy of the system is

$$\Phi(\theta) = V(0) + \frac{1}{2} C r_0^2 + \frac{1}{2} \frac{(k - C r_0 \cos\theta)^2}{(A + B_1)\sin^2\theta + C_1 \cos^2\theta + J} \tag{11}$$

The investigated stationary motion satisfies the condition

$$(\partial\Phi / \partial\theta)_0 = 0 \tag{12}$$

If the constants k, r_0 are not perturbed, then, by Routh's theorem, the motion (5) is stable with respect to θ, $\dot{\theta}$, if in addition the function $\phi(\theta)$ in the unperturbed motion has a minimum, that is

$$(\partial^2\Phi / \partial\theta^2)_0 > 0 \tag{13}$$

Using (12) we can put the condition (13) in the form

$$C^2\omega^2 \sin^2\theta_0 + [(A + B_1)\sin^2\theta_0 + C_1\cos^2\theta_0 + J]\,[V''(\theta_0) - V'(\theta_0)\operatorname{ctg}\theta_0 + \\ + (A + B_1 - C_1)\Omega^2\sin^2\theta_0] + 4(A + B_1 - C_1)V'(\theta_0)\sin\theta_0\cos\theta_0 > 0 \tag{14}$$

In the above inequality it is essential that $\sin\theta_0 \neq 0$.

In the case when $(\partial^2\phi\, \partial\theta^2)_0 < 0$ the motion (5) is unstable [6]; consequently the condition (14) is the necessary and sufficient condition (excluding boundary) of stability of the regular precession of a gyroscope on gimbals. The conditions (7), (9), and (14) in the case of a heavy gyroscope on gimbals with vertical axis of the outer ring ($V(\theta) = a\cos\theta$) reduce to the conditions obtained for similar cases in [2, 4]. The necessity of the condition (14) for a gyroscope not on gimbals ($A_1 = B_1 = C_1 = J = 0$) has been demonstrated in [5].

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